Spin Orbit Spectra Calculations

```
(%i1) kill(all);
(%o0) done
```

1 Define operators

```
(%i1) assume(h[bar]>0, m>0, a[0]>0, b>0, Z>0);
```

(%o1) $[h_{bar}>0, m>0, a_0>0, b>0, Z>0]$

```
(%i2) /* Norm of radial function */
N(f) := (integrate(conjugate(f)*f*r^2, r, 0, inf));
```

(%o2) $N(f):=\int_0^{\infty}\mathrm{conjugate}(f)\,f\,r^2\,dr$

```
(%i3) /* Norm of spherical function */
NY(f) := integrate(integrate(conjugate(f)*f*sin(theta), theta, 0, %pi
```

(%o3) $NY(f):=\int_0^{2\pi}\int_0^{\pi}\mathrm{conjugate}(f)\,f\,\sin(\theta)\,d\theta\,d\phi$

```
(%i4) /* Norm of 3d function */
N3(f) := integrate(integrate(integrate(conjugate(f)*f*sin(theta), the
phi, 0, 2*%pi)*r^2, r, 0, inf);
```

(%o4) $N3(f):=\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}\mathrm{conjugate}(f)\,f\,\sin(\theta)\,d\theta\,d\phi\,r^2\,dr$

```
(%i5) /* Expectation value of radial function */
Ex(f,op) := (integrate(conjugate(f)*op*f*r^2, r, 0, inf));
```

(%o5) $Ex(f,op):=\int_0^{\infty}\mathrm{conjugate}(f)\,op\,f\,r^2\,dr$

```
(%i6) /* Expectation value of 3D wave function */
Ex3(f,op) := integrate(integrate(integrate(conjugate(f)*op*f*sin(thet
phi, 0, 2*%pi)*r^2, r, 0, inf);
```

(%o6) $Ex3(f,op):=\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}\mathrm{conjugate}(f)\,op\,f\,\sin(\theta)\,d\theta\,d\phi\,r^2\,dr$

```
(%i7) /* Integral of two 3D wave functions */
Ex32(f1,op,f2) := integrate(integrate(integrate(conjugate(f1)*op*f2*s
phi, 0, 2*%pi)*r^2, r, 0, inf);
```

(%o7) $Ex32(f1,op,f2):=\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}\mathrm{conjugate}(f1)\,op\,f2\,\sin(\theta)\,d\theta\,d\phi\,r^2\,dr$

Define energy levels of Hydrogen

2 Define Radial Eigenfunctions

```
(%i8) rhon: 2*Z*r/(n*a[0]);
```
(%o8) $\frac{2\,r\,Z}{a_0\,n}$

1s radial function

```
(%i9) rho: ev(rhon, [n=1]);
```
(%o9) $\frac{2\,r\,Z}{a_0}$

```
(%i10) R[0]: 2*(Z/a[0])^(3/2)*exp(-rho/2);
```
(%o10) $\frac{2\,Z^{3/2}\,\%e^{-\frac{r\,Z}{a_0}}}{a_0^{3/2}}$

2s radial function

```
(%i11) rho: ev(rhon, [n=2]);
```
(%o11) $\frac{r\,Z}{a_0}$

```
(%i12) R[1]: 1/(2*sqrt(2))*(Z/a[0])^(3/2)*(2-rho)*exp(-rho/2);
```
(%o12) $\frac{Z^{3/2}\left(2-\frac{r\,Z}{a_0}\right)\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{3/2}\,a_0^{3/2}}$

2p radial function

```
(%i13) R[2]: 1/(2*sqrt(6))*(Z/a[0])^(3/2)*(rho)*exp(-rho/2);
```
(%o13) $\frac{r\,Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2\sqrt{6}\,a_0^{5/2}}$

3s radial function

```
(%i14) rho: ev(rhon, [n=3]);
```
(%o14) $\frac{2\,r\,Z}{3\,a_0}$

```
(%i15) R[3]: 1/(sqrt(243))*(Z/a[0])^(3/2)*(6-6*rho+rho^2)*exp(-rho/2);
```
(%o15) $\frac{Z^{3/2}\left(\frac{4\,r^2\,Z^2}{9\,a_0^2}-\frac{4\,r\,Z}{a_0}+6\right)\%e^{-\frac{r\,Z}{3\,a_0}}}{3^{5/2}\,a_0^{3/2}}$

3p radial function

```
(%i16) R[4]: 1/(sqrt(486))*(Z/a[0])^(3/2)*(4-rho)*rho*exp(-rho/2);
```

(%o16) $$\frac{2\,r\,Z^{5/2}\left(4-\frac{2\,r\,Z}{3\,a_0}\right)\%e^{-\frac{r\,Z}{3\,a_0}}}{27\,\sqrt{6}\,a_0^{5/2}}$$

3d radial function

```
(%i17) R[5]: 1/(sqrt(2430))*(Z/a[0])^(3/2)*(rho^2)*exp(-rho/2);
```

(%o17) $$\frac{4\,r^2\,Z^{7/2}\,\%e^{-\frac{r\,Z}{3\,a_0}}}{81\,\sqrt{30}\,a_0^{7/2}}$$

Normalization check

```
(%i18) for i: 0 thru 5 do (
       print (i, "  N(R): ", N(R[i]))
       );
```

0 *N(R):* 1
1 *N(R):* 1
2 *N(R):* 1
3 *N(R):* 1
4 *N(R):* 1
5 *N(R):* 1
(%o18) *done*

3 Spherial Harmonics

Define Eigenfunctions

Y(0,0)

```
(%i19) Y[0]: 1/(2*sqrt(%pi));
```

(%o19) $$\frac{1}{2\sqrt{\pi}}$$

Y(1,0)

```
(%i20) Y[1]: 1/2*sqrt(3/%pi)*cos(theta);
```

(%o20) $$\frac{\sqrt{3}\,\cos(\theta)}{2\sqrt{\pi}}$$

Y(1,1)

(%i21) Y[2]: -1/2*sqrt(3/(2*%pi))*sin(theta)*exp(%i*phi);

(%o21) $-\frac{\sqrt{3}\,\%e^{\%i\,\phi}\sin(\theta)}{2^{3/2}\sqrt{\pi}}$

Y(2,0)

(%i22) Y[3]: 1/4*sqrt(5/%pi)*(3*cos(theta)^2-1);

(%o22) $\frac{\sqrt{5}\,(3\cos(\theta)^2-1)}{4\sqrt{\pi}}$

Y(2,1)

(%i23) Y[4]: -1/2*sqrt(15/(2*%pi))*sin(theta)*cos(theta)*exp(%i*phi);

(%o23) $-\frac{\sqrt{15}\,\%e^{\%i\,\phi}\cos(\theta)\sin(\theta)}{2^{3/2}\sqrt{\pi}}$

Y(2,2)

(%i24) Y[5]: 1/4*sqrt(15/(2*%pi))*sin(theta)^2*exp(2*%i*phi);

(%o24) $\frac{\sqrt{15}\,\%e^{2\,\%i\,\phi}\sin(\theta)^2}{2^{5/2}\sqrt{\pi}}$

Y(3,0)

(%i25) Y[6]: 1/4*sqrt(7/%pi)*(5*cos(theta)^3-3*cos(theta));

(%o25) $\frac{\sqrt{7}\,(5\cos(\theta)^3-3\cos(\theta))}{4\sqrt{\pi}}$

Y(3,1)

(%i26) Y[7]: -1/8*sqrt(21/(%pi))*sin(theta)*(5*cos(theta)^2-1)*exp(%i*phi);

(%o26) $-\frac{\sqrt{21}\,\%e^{\%i\,\phi}(5\cos(\theta)^2-1)\sin(\theta)}{8\sqrt{\pi}}$

Y(3,2)

(%i27) Y[8]: 1/4*sqrt(105/(2*%pi))*sin(theta)^2*cos(theta)*exp(2*%i*phi);

(%o27) $\frac{\sqrt{105}\,\%e^{2\,\%i\,\phi}\cos(\theta)\sin(\theta)^2}{2^{5/2}\sqrt{\pi}}$

Y(3,3)

```
(%i28) Y[9]: -1/8*sqrt(35/(%pi))*sin(theta)^3*exp(3*%i*phi);
```

(%o28) $-\frac{\sqrt{35}\,\%e^{3\,\%i\,\phi}\,\sin(\theta)^3}{8\,\sqrt{\pi}}$

4 Wave functions psi(r,theta,phi)=R[n]*Y[l,m] and radial derivatives dpsi=dR[n]*Y[l,m]

psi[n=1, l=0, ml=0]

```
(%i29) qn[0]: "n=1, l=0, ml=0"$
```

```
(%i30) psi[0]:  R[0]*Y[0]$
```

psi[n=2, l=0, ml=0]

```
(%i31) qn[1]: "n=2, l=0, ml=0"$
```

```
(%i32) psi[1]:  R[1]*Y[0]$
```

psi[n=2, l=1, ml=0]

```
(%i33) qn[2]: "n=2, l=1, ml=0"$
```

```
(%i34) psi[2]:  R[2]*Y[1]$
```

psi[n=2, l=1, ml=1]

```
(%i35) qn[3]: "n=2, l=1, ml=1"$
```

```
(%i36) psi[3]:  R[2]*Y[2]$
```

psi[n=3, l=0, ml=0]

```
(%i37) qn[4]: "n=3, l=0, ml=0"$
```

```
(%i38) psi[4]:  R[3]*Y[0]$
```

psi[n=3, l=1, ml=0]

```
(%i39) qn[5]: "n=3, l=1, ml=0"$
```

```
(%i40) psi[5]:  R[4]*Y[1]$
```

psi[n=3, l=1, ml=1]

```
(%i41) qn[6]: "n=3, l=1, ml=1"$
```

```
(%i42) psi[6]:  R[4]*Y[2]$
```

psi[n=3, l=2, ml=0]

```
(%i43) qn[7]: "n=3, l=2, ml=0"$
```

```
(%i44) psi[7]:  R[5]*Y[3]$
```

psi[n=3, l=2, ml=1]

```
(%i45) qn[8]: "n=3, l=2, ml=1"$
```

```
(%i46) psi[8]:  R[5]*Y[4]$
```

psi[n=3, l=2, ml=2]

```
(%i47) qn[9]: "n=3, l=2, ml=2"$
```

```
(%i48) psi[9]:  R[5]*Y[5]$
```

```
(%i49) for i: 0 thru 9 do (
           print (qn[i], ", psi: ", psi[i])
       );
```

n=1, l=0, ml=0 , psi: $\dfrac{Z^{3/2}\,\%e^{-\frac{r\,Z}{a_0}}}{\sqrt{\pi}\,a_0^{3/2}}$

n=2, l=0, ml=0 , psi: $\dfrac{Z^{3/2}\left(2-\frac{r\,Z}{a_0}\right)\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\sqrt{\pi}\,a_0^{3/2}}$

n=2, l=1, ml=0 , psi: $\dfrac{\sqrt{3}\,r\cos(\theta)\,Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{4\sqrt{6}\sqrt{\pi}\,a_0^{5/2}}$

n=2, l=1, ml=1 , psi: $-\dfrac{\sqrt{3}\,\%e^{\%i\,\phi}\,r\sin(\theta)\,Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\sqrt{6}\sqrt{\pi}\,a_0^{5/2}}$

n=3, l=0, ml=0 , psi: $\dfrac{Z^{3/2}\left(\frac{4\,r^2\,Z^2}{9\,a_0^2}-\frac{4\,r\,Z}{a_0}+6\right)\%e^{-\frac{r\,Z}{3\,a_0}}}{2\;3^{5/2}\sqrt{\pi}\,a_0^{3/2}}$

n=3, l=1, ml=0 , psi: $\dfrac{r\cos(\theta)\,Z^{5/2}\left(4-\frac{2\,r\,Z}{3\,a_0}\right)\%e^{-\frac{r\,Z}{3\,a_0}}}{3^{5/2}\sqrt{6}\sqrt{\pi}\,a_0^{5/2}}$

n=3, l=1, ml=1 , psi: $-\dfrac{\%e^{\%i\,\phi}\,r\sin(\theta)\,Z^{5/2}\left(4-\frac{2\,r\,Z}{3\,a_0}\right)\%e^{-\frac{r\,Z}{3\,a_0}}}{\sqrt{2}\;3^{5/2}\sqrt{6}\sqrt{\pi}\,a_0^{5/2}}$

n=3, l=2, ml=0 , psi: $\dfrac{\sqrt{5}\,r^2\left(3\cos(\theta)^2-1\right)Z^{7/2}\,\%e^{-\frac{r\,Z}{3\,a_0}}}{81\sqrt{30}\sqrt{\pi}\,a_0^{7/2}}$

n=3, l=2, ml=1 , psi: $-\dfrac{\sqrt{2}\sqrt{15}\,\%e^{\%i\,\phi}\,r^2\cos(\theta)\sin(\theta)\,Z^{7/2}\,\%e^{-\frac{r\,Z}{3\,a_0}}}{81\sqrt{30}\sqrt{\pi}\,a_0^{7/2}}$

n=3, l=2, ml=2 , psi: $\dfrac{\sqrt{15}\,\%e^{2\,\%i\,\phi}\,r^2\sin(\theta)^2\,Z^{7/2}\,\%e^{-\frac{r\,Z}{3\,a_0}}}{81\sqrt{2}\sqrt{30}\sqrt{\pi}\,a_0^{7/2}}$

(%o49) *done*

Normalization check

```
(%i50) for i: 0 thru 9 do
       print (i, "  N3(psi): ", N3(psi[i]));
0  N3(psi):  1
1  N3(psi):  1
2  N3(psi):  1
3  N3(psi):  1
4  N3(psi):  1
5  N3(psi):  1
6  N3(psi):  1
7  N3(psi):  1
8  N3(psi):  1
9  N3(psi):  1
(%o50) done
```

5 Expectation values of orbital velocities

```
(%i51) assume(epsilon>0, epsilon<1);
```

(%o51) $[\varepsilon>0,\varepsilon<1]$

```
(%i52) op: a[0]/(1+epsilon*cos(theta));
```

(%o52) $\frac{a_0}{\varepsilon\cos(\theta)+1}$

5.1 Detailed results

```
(%i53) for i: 0 thru 9 do (
           P: r*psi[i],
           d2P: diff(P, r, 2),
           v2: expand(ratsimp(-h[bar]^2/P * d2P)),
           v: Ex3(psi[i], op),
           print ("******************************************************"),
           print (qn[i], ", psi: ", psi[i]),
           print ("***** d2P: ", d2P),
           print ("***** v2: ", v2)
       );
```

****

n=1, l=0, ml=0 , psi: $\frac{Z^{3/2}\,\%e^{-\frac{r\,Z}{a_0}}}{\sqrt{\pi}\,a_0^{3/2}}$

***** d2P: $\frac{r\,Z^{7/2}\,\%e^{-\frac{r\,Z}{a_0}}}{\sqrt{\pi}\,a_0^{7/2}}-\frac{2\,Z^{5/2}\,\%e^{-\frac{r\,Z}{a_0}}}{\sqrt{\pi}\,a_0^{5/2}}$

***** v2: $\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{h_{bar}^2\,Z^2}{a_0^2}$

****

n=2, l=0, ml=0 , psi: $\frac{Z^{3/2}\left(2-\frac{r\,Z}{a_0}\right)\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\,\sqrt{\pi}\,a_0^{3/2}}$

***** d2P: $\frac{r\,Z^{7/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\,\sqrt{\pi}\,a_0^{7/2}}-\frac{Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{3/2}\,\sqrt{\pi}\,a_0^{5/2}}+\frac{r\,Z^{7/2}\left(2-\frac{r\,Z}{a_0}\right)\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{9/2}\,\sqrt{\pi}\,a_0^{7/2}}-\frac{Z^{5/2}\left(2-\frac{r\,Z}{a_0}\right)\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\,\sqrt{\pi}\,a_0^{5/2}}$

***** v2: $\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{h_{bar}^2\,Z^2}{4\,a_0^2}$

****

n=2, l=1, ml=0 , psi: $\frac{\sqrt{3}\,r\cos(\theta)Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{4\sqrt{6}\,\sqrt{\pi}\,a_0^{5/2}}$

***** d2P: $\frac{\sqrt{3}\,r^2\cos(\theta)Z^{9/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{16\sqrt{6}\,\sqrt{\pi}\,a_0^{9/2}}-\frac{\sqrt{3}\,r\cos(\theta)Z^{7/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2\sqrt{6}\,\sqrt{\pi}\,a_0^{7/2}}+\frac{\sqrt{3}\cos(\theta)Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2\sqrt{6}\,\sqrt{\pi}\,a_0^{5/2}}$

***** v2: $-\frac{h_{bar}^2\,Z^2}{4\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{2\,h_{bar}^2}{r^2}$

****

n=2, l=1, ml=1 , psi: $-\frac{\sqrt{3}\,\%e^{\%i\,\phi}\,r\sin(\theta)Z^{5/2}\,\%e^{-\frac{r\,Z}{2\,a_0}}}{2^{5/2}\sqrt{6}\,\sqrt{\pi}\,a_0^{5/2}}$

***** d2P: $-\frac{\sqrt{3}\,r^2\sin(\theta)Z^{9/2}\,\%e^{\%i\,\phi-\frac{r\,Z}{2\,a_0}}}{2^{9/2}\sqrt{6}\,\sqrt{\pi}\,a_0^{9/2}}+\frac{\sqrt{3}\,r\sin(\theta)Z^{7/2}\,\%e^{\%i\,\phi-\frac{r\,Z}{2\,a_0}}}{2^{3/2}\sqrt{6}\,\sqrt{\pi}\,a_0^{7/2}}-$
$\frac{\sqrt{3}\sin(\theta)Z^{5/2}\,\%e^{\%i\,\phi-\frac{r\,Z}{2\,a_0}}}{2^{3/2}\sqrt{6}\,\sqrt{\pi}\,a_0^{5/2}}$

5.2 Compacted results

```
(%i54) for i: 0 thru 9 do (
           P: r*psi[i],
           d2P: diff(P, r, 2),
           v2: expand(ratsimp(-h[bar]^2/P * d2P)),
           v: Ex3(psi[i], op),
           print ("*******************************************************"),
           print (qn[i], ", v2: ", v2)
       );
```

****

$n{=}1,\ l{=}0,\ ml{=}0,\ v2{:}\ \frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{h_{bar}^2\,Z^2}{a_0^2}$

****

$n{=}2,\ l{=}0,\ ml{=}0,\ v2{:}\ \frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{h_{bar}^2\,Z^2}{4\,a_0^2}$

****

$n{=}2,\ l{=}1,\ ml{=}0,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{4\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{2\,h_{bar}^2}{r^2}$

****

$n{=}2,\ l{=}1,\ ml{=}1,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{4\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{2\,h_{bar}^2}{r^2}$

****

$n{=}3,\ l{=}0,\ ml{=}0,\ v2{:}\ \frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{h_{bar}^2\,Z^2}{9\,a_0^2}$

****

$n{=}3,\ l{=}1,\ ml{=}0,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{9\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{2\,h_{bar}^2}{r^2}$

****

$n{=}3,\ l{=}1,\ ml{=}1,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{9\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{2\,h_{bar}^2}{r^2}$

****

$n{=}3,\ l{=}2,\ ml{=}0,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{9\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{6\,h_{bar}^2}{r^2}$

****

$n{=}3,\ l{=}2,\ ml{=}1,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{9\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{6\,h_{bar}^2}{r^2}$

****

$n{=}3,\ l{=}2,\ ml{=}2,\ v2{:}\ -\frac{h_{bar}^2\,Z^2}{9\,a_0^2}+\frac{2\,h_{bar}^2\,Z}{a_0\,r}-\frac{6\,h_{bar}^2}{r^2}$

(%o54) *done*

6 Expectation values of orbital radii in theta pl (see notation of Atkins)

```
(%i55) op: a[0]/(1+epsilon*cos(theta));
```

(%o55) $$\frac{a_0}{\varepsilon \cos(\theta)+1}$$

```
(%i56) for i: 0 thru 9 do (
          re: Ex3(psi[i], op),
          print ("************************************************************"),
          print (qn[i], ", <r>: ", re)
       );
```

****

n=1, l=0, ml=0 , <r>: $\frac{a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{\log(1-\varepsilon)}{\varepsilon}\right)}{2}$

****

n=2, l=0, ml=0 , <r>: $\frac{a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{\log(1-\varepsilon)}{\varepsilon}\right)}{2}$

****

n=2, l=1, ml=0 , <r>: $\frac{3\,a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon^3}-\frac{\varepsilon+2}{2\,\varepsilon^2}+\frac{\varepsilon-2}{2\,\varepsilon^2}-\frac{\log(1-\varepsilon)}{\varepsilon^3}\right)}{2}$

****

n=2, l=1, ml=1 , <r>: $\frac{3\,a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{\log(\varepsilon+1)}{\varepsilon^3}+\frac{\varepsilon+2}{2\,\varepsilon^2}-\frac{\log(1-\varepsilon)}{\varepsilon}-\frac{\varepsilon-2}{2\,\varepsilon^2}+\frac{\log(1-\varepsilon)}{\varepsilon^3}\right)}{4}$

****

n=3, l=0, ml=0 , <r>: $\frac{a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{\log(1-\varepsilon)}{\varepsilon}\right)}{2}$

****

n=3, l=1, ml=0 , <r>: $\frac{3\,a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon^3}-\frac{\varepsilon+2}{2\,\varepsilon^2}+\frac{\varepsilon-2}{2\,\varepsilon^2}-\frac{\log(1-\varepsilon)}{\varepsilon^3}\right)}{2}$

****

n=3, l=1, ml=1 , <r>: $\frac{3\,a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{\log(\varepsilon+1)}{\varepsilon^3}+\frac{\varepsilon+2}{2\,\varepsilon^2}-\frac{\log(1-\varepsilon)}{\varepsilon}-\frac{\varepsilon-2}{2\,\varepsilon^2}+\frac{\log(1-\varepsilon)}{\varepsilon^3}\right)}{4}$

****

n=3, l=2, ml=0 , <r>: $(5\,a_0\,(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{6\log(\varepsilon+1)}{\varepsilon^3}+\frac{9\log(\varepsilon+1)}{\varepsilon^5}+$
$\frac{3\,\varepsilon^3+12\,\varepsilon^2-18\,\varepsilon-36}{4\,\varepsilon^4}-\frac{3\,\varepsilon^3-12\,\varepsilon^2-18\,\varepsilon+36}{4\,\varepsilon^4}-\frac{\log(1-\varepsilon)}{\varepsilon}+\frac{6\log(1-\varepsilon)}{\varepsilon^3}-\frac{9\log(1-\varepsilon)}{\varepsilon^5}))/8$

****

n=3, l=2, ml=1 , <r>:
$\frac{15\,a_0\left(\frac{\log(\varepsilon+1)}{\varepsilon^3}-\frac{\log(\varepsilon+1)}{\varepsilon^5}-\frac{3\,\varepsilon^3+8\,\varepsilon^2-6\,\varepsilon-12}{12\,\varepsilon^4}+\frac{3\,\varepsilon^3-8\,\varepsilon^2-6\,\varepsilon+12}{12\,\varepsilon^4}-\frac{\log(1-\varepsilon)}{\varepsilon^3}+\frac{\log(1-\varepsilon)}{\varepsilon^5}\right)}{4}$

****

n=3, l=2, ml=2 , <r>: $(15\,a_0\,(\frac{\log(\varepsilon+1)}{\varepsilon}-\frac{2\log(\varepsilon+1)}{\varepsilon^3}+\frac{\log(\varepsilon+1)}{\varepsilon^5}+\frac{9\,\varepsilon^3+20\,\varepsilon^2-6\,\varepsilon-12}{12\,\varepsilon^4}$
$-\frac{9\,\varepsilon^3-20\,\varepsilon^2-6\,\varepsilon+12}{12\,\varepsilon^4}-\frac{\log(1-\varepsilon)}{\varepsilon}+\frac{2\log(1-\varepsilon)}{\varepsilon^3}-\frac{\log(1-\varepsilon)}{\varepsilon^5}))/16$

7 Expectation values of orbital radii in phi plan (see notation of Atkins)

```
(%i57) op: a[0]/(1+epsilon*cos(phi));
```

(%o57) $$\frac{a_0}{\varepsilon \cos(\phi)+1}$$

```
(%i58) for i: 0 thru 9 do (
           re: Ex3(psi[i], op),
           print ("*******************************************************"),
           print (qn[i], ", <r>: ", re)
       );
```

Is $\sqrt{1-\varepsilon^2}-\varepsilon+1$ *positive, negative, or zero?* p;

Is $\left|\sqrt{1-\varepsilon^2}-1\right|-\varepsilon$ *positive, negative, or zero?* n;

****

n=1, l=0, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=2, l=0, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=2, l=1, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=2, l=1, ml=1 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=0, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=1, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=1, ml=1 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=2, ml=0 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=2, ml=1 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

****

n=3, l=2, ml=2 , <r>: $\frac{a_0}{\sqrt{1-\varepsilon^2}}$

(%o58) *done*