272(2): Transition to Precessing Planar Ellipse

The beta ellipse is:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(1)$$

with

$$\tan\beta = \frac{L_Z}{L}\tan\phi \quad -(2)$$

Therefore: $$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{(1-\cos^2\beta)^{1/2}}{\cos\beta} = \frac{L_Z}{L}\tan\phi \quad -(3)$$

i.e. $$\cos^2\beta = \frac{1}{1+\left(\frac{L_Z}{L}\right)^2\tan^2\phi} \quad -(4)$$

$$= \frac{\cos^2\phi}{\cos^2\phi+\left(\frac{L_Z}{L}\right)^2\sin^2\phi}$$

It is clear that as:

$$L \rightarrow L_Z \quad -(5)$$

$$\beta \rightarrow \phi \quad -(6)$$

Now define:

$$\cos(x\phi) = \frac{\cos\phi}{\left(\cos^2\phi+\left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}} \quad -(7)$$

so

$$x = \frac{1}{\phi}\cos^{-1}\left(\frac{\cos\phi}{\left(\cos^2\phi+\left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}}\right) \quad -(8)$$

2) In general therefore the 3-D theory gives the processing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\phi)} \quad -(9)$$

with x defined by Eq. (8).

It is observed experimentally that:

$$x = 1 + \frac{3MG}{\alpha c^2} \quad -(10)$$

so eqs. (8) and (10) determine an angle ϕ_0:

$$\frac{1}{\phi_0}\cos^{-1}\left(\frac{\cos\phi_0}{\left(\cos^2\phi_0+\left(\frac{L_Z}{L}\right)^2\sin^2\phi_0\right)^{1/2}}\right) = 1+\frac{3MG}{\alpha c^2} \quad -(11)$$

Therefore three dimensional orbits have evolved to processing two dimensional orbits, which process.

In general however:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(12)$$

where

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi+\left(\frac{L_Z}{L}\right)^2\sin^2\phi\right)^{1/2}} \quad -(13)$$

It is clear that eq. (12) is no longer an ellipse. Therefore primordial three dimensional orbits evolve into the observed two dimensional precessing orbits. The time dependence of this process is given by:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(14)$$

so

$$dt = \frac{mr^2}{L} d\beta \quad -(15)$$

and

$$t = \frac{m\alpha^2}{L} \int \frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(16)$$

so t can be evaluated in terms of β and of ϕ. To evaluate eq. (16) we use Kepler's equation

$$\frac{2\pi t}{\tau} = \phi - \epsilon \sin\phi \quad -(17)$$

for the mean anomaly, where:

$$\tan\frac{\beta}{2} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/2} \tan\frac{\phi}{2} \quad -(18)$$

Here τ is the time taken to sweep out an entire orbit.

Now use the half angle formula:

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \quad -(19)$$

So:

$$\frac{1-\cos\beta}{1+\cos\beta} = \left(\frac{1+\epsilon}{1-\epsilon}\right)\left(\frac{1-\cos\phi}{1+\cos\phi}\right) \quad -(20)$$

where:

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(21)$$

so ϕ and $\sin\phi$ can be found in terms of $\cos\phi$, and t evaluated.

The quantity $\cos\beta$ can also be expressed in terms of θ:

$$\cos^2\beta = 1 - \frac{L^2}{L^2 - L_z^2}\cos^2\theta \quad -(22)$$

i.e.

$$(L^2 - L_z^2)\cos^2\beta = L^2(1-\cos^2\theta) - L_z^2 \quad -(23)$$

$$= (L^2 - L_z^2)\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi}$$

It is clear that θ is a function of ϕ and r can be expressed both in terms of ϕ and in terms of θ. Eq. (23) removes any restriction due to the use of a tan function.

5) Suggested Graphics

1) Plot the function:

$$r = \frac{\alpha}{1 + \epsilon \dfrac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi\right)^{1/2}}} \quad -(24)$$

and compare it directly with:

$$r = \frac{\alpha}{1 + \epsilon \cos\phi} \quad -(25)$$

2) Plot the function:

$$x = \frac{1}{\phi} \cos^{-1}\left(\frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_Z}{L}\right)^2 \sin^2\phi\right)^{1/2}}\right) \quad -(26)$$

3) Plot and animate t versus ϕ from eq. (16).
