435(3): Splittings and Shifts Produced by Frame Transformation

In frame (r_1, θ, ϕ) the classical hamiltonian is:

$$H_1 = \frac{p_1^2}{2m} + U_1 \quad -(1)$$

and the time dependent Schroedinger equation is:

$$i\hbar \frac{d\psi_1}{dt_1} = E_1 \psi_1 = H_1 \psi_1 \quad -(2)$$

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The transformation from (r_1, θ, ϕ) to (r, θ, ϕ) is carried out with:

$$\left. \begin{array}{l} r_1 = \dfrac{r}{m(r)^{1/2}}, \quad \underline{p_1} = \dfrac{p}{m(r)^{1/2}}, \\ \psi_1 \rightarrow \psi, \quad \nabla_1^2 \rightarrow \nabla^2 \end{array} \right\} -(5)$$

$$dt_1 \rightarrow m(r)^{1/2} dt$$

so eq. (2) becomes:

$$\boxed{\frac{i\hbar}{m(r)^{1/2}} \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \nabla^2 \left(\frac{\psi}{m(r)} \right) - \frac{e^2}{4\pi\epsilon_0} \frac{m(r)^{1/2}}{r} \psi} \quad -(6)$$

This equation produces shifts and splittings of spectra.

In the limit:

$$m(r) \rightarrow 1 \quad -(7)$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi - \frac{e^2}{4\pi \epsilon_0 r} \psi \quad -(8)$$

Assume that the time dependent wave function ψ is:

$$\psi = e^{i\omega t} \psi_H (r, \theta, \phi) \quad -(9)$$

where $\psi_H(r, \theta, \phi)$ is the hydrogenic wave function. It follows that:

$$i\hbar \int \psi^* \frac{\partial \psi}{\partial t} d\tau = -\hbar \omega \quad -(10)$$

and

$$-\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \psi \, d\tau - \frac{e^2}{4\pi \epsilon_0} \int \psi^* \frac{1}{r} \psi \, d\tau$$

$$= -\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(11)$$

so the energy levels of the H atom are:

$$E = -\hbar \omega = -\left(\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \right) \frac{1}{n^2} \quad -(12)$$

i.e.

$$\boxed{\hbar \omega_n = \left(\frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \right) \frac{1}{n^2}} \quad -(13)$$

From eq. (6) this result is generalized to:

$$i\hbar \int \psi^* \frac{1}{m(r)^{1/2}} \frac{\partial \psi}{\partial t} d\tau = -\frac{\hbar^2}{2m} \int \psi^* \nabla^2 \left(\frac{\psi}{m(r)} \right) d\tau - \frac{e^2}{4\pi \epsilon_0} \int \psi^* \frac{m(r)^{1/2}}{r} \psi \, d\tau$$

$$-(14)$$

The number of new energy levels produced by

3) both sides of eq. (14) must be the same, so to work out the energy levels, it is sufficient to compute the left hand side of eq. (14). The same $n(r)$ must be used on both sides of eq. (14). The new energy levels of the H atom describe the Lamb shift due to the effect of the vacuum.