

DEFINITIVE PROOF 2: THE FUNDAMENTAL ORIGIN OF TORSION AND CURVATURE

The torsion and curvature tensors in general relativity are defined by the action of the commutator of covariant derivatives on any tensor. This proof considers the action of the commutator on the vector in any dimension and in any spacetime. The proof is true irrespective of any assumption, even fundamentals such as metric compatibility or tetrad postulate. The spacetime torsion is always present in general relativity and cannot be ignored. This means that the connection is always antisymmetric, not symmetric as in the standard model.

Proof:

Consider the commutator of covariant derivatives, and let it operate on the vector V^ρ in any dimension in any spacetime. Thus:

$$[D_\mu, D_\nu] V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad (1)$$

On the right hand side, the covariant derivatives act on other covariant derivatives contained within the brackets. These are defined by:

$$D_\nu V^\rho = \partial_\nu V^\rho + \Gamma_{\nu\lambda}^\rho V^\lambda \quad (2)$$

$$D_\lambda V^\rho = \partial_\lambda V^\rho + \Gamma_{\lambda\sigma}^\rho V^\sigma \quad (3)$$

$$D_\nu V^\sigma = \partial_\nu V^\sigma + \Gamma_{\nu\lambda}^\sigma V^\lambda \quad (4)$$

in Riemann geometry and are regarded as tensors of rank two. The covariant derivative of a tensor of rank two must therefore be used in Eq. (1). Thus,

$$\begin{aligned} [D_\mu, D_\nu] V^\rho &= \partial_\mu (D_\nu V^\rho) - \Gamma_{\mu\nu}^\lambda D_\lambda V^\rho + \Gamma_{\mu\sigma}^\rho D_\nu V^\sigma - \partial_\nu (D_\mu V^\rho) \\ &\quad + \Gamma_{\nu\mu}^\lambda D_\lambda V^\rho - \Gamma_{\nu\sigma}^\rho D_\mu V^\sigma \end{aligned} \quad (5)$$

Therefore, there are equations such as:

$$\begin{aligned} \partial_\mu (D_\nu V^\rho) &= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma_{\nu\lambda}^\rho) V^\lambda + \Gamma_{\nu\lambda}^\rho \partial_\mu V^\lambda \\ &= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma_{\nu\sigma}^\rho) V^\sigma + \Gamma_{\nu\sigma}^\rho \partial_\mu V^\sigma \end{aligned} \quad (6)$$

The summation indices (dummy indices) are now rearranged as follows:

$$\lambda \rightarrow \sigma \quad (7)$$

This procedure gives:

$$\begin{aligned} [D_\mu, D_\nu] V^\rho = & \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma_{\nu\sigma}^\rho) V^\sigma + \Gamma_{\nu\sigma}^\rho \partial_\mu V^\sigma \\ & - \Gamma_{\mu\nu}^\lambda \partial_\lambda V^\rho - \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\rho V^\sigma \\ & + \Gamma_{\mu\sigma}^\rho \partial_\nu V^\sigma + \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\lambda}^\sigma V^\lambda \\ & - \partial_\nu \partial_\mu V^\rho - (\partial_\nu \Gamma_{\mu\sigma}^\rho) V^\sigma - \Gamma_{\mu\sigma}^\rho \partial_\nu V^\sigma \\ & + \Gamma_{\nu\mu}^\lambda \partial_\lambda V^\rho + \Gamma_{\nu\mu}^\lambda \Gamma_{\lambda\sigma}^\rho V^\sigma \\ & - \Gamma_{\mu\sigma}^\rho \partial_\nu V^\sigma - \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\lambda}^\sigma V^\lambda \end{aligned} \quad (8)$$

This expression is re-arranged to give:

$$\begin{aligned} [D_\mu, D_\nu] V^\rho = & (\partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda) V^\sigma \\ & - (\Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda) (\partial_\lambda V^\rho + \Gamma_{\lambda\sigma}^\rho V^\sigma) \end{aligned} \quad (9)$$

which is expressed as:

$$[D_\mu, D_\nu] V^\rho = R_{\sigma\mu\nu}^\rho V^\sigma - T_{\mu\nu}^\lambda D_\lambda V^\rho \quad (10)$$

In this equation appears the curvature tensor:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \quad (11)$$

and the torsion tensor:

$$T_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \quad (12)$$

Therefore, quod erat demonstratum:

$$[D_\mu, D_\nu] V^\rho = R_{\sigma\mu\nu}^\rho V^\sigma - T_{\mu\nu}^\lambda D_\lambda V^\rho \quad (13)$$

