195(2): Derivation of the Orbital Equation from the Crothers Metric.

The starting lagrangian is:

$$\mathcal{L} = \frac{1}{2} mc^2 = \frac{m}{2}\left(c^2 A C^{1/2}(r) \left(\frac{dt}{d\tau}\right)^2 - B C^{1/2}(r)\left(\frac{dr}{d\tau}\right)^2 - C(r)\left(\frac{d\theta}{d\tau}\right)^2 \right) \quad -(1)$$

The total energy is:

$$E = mc^2 A C^{1/2}(r) \frac{dt}{d\tau} \quad -(2)$$

and the total angular momentum is:

$$L = m C(r) \frac{d\theta}{d\tau} \quad -(3)$$

So:

$$E^2 = m^2 c^4 A^2 C(r) \left(\frac{dt}{d\tau}\right)^2 \quad -(4)$$

$$L^2 = m^2 C^2(r) \left(\frac{d\theta}{d\tau}\right)^2 \quad -(5)$$

Therefore:

$$mc^2 = \frac{1}{A C^{1/2}(r)} \frac{E^2}{mc^2} - m B C^{1/2}(r)\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{m}\frac{L^2}{C(r)} \quad -(6)$$

$$\frac{mc^2}{B C^{1/2}(r)} = \frac{1}{ABC(r)} \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{m C^{3/2}(r) B} \quad -(7)$$

2)

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{ABC(r)}\frac{E^2}{mc^2} - \frac{mc^2}{BC^{1/2}(r)} - \frac{L^2}{mBC^{3/2}(r)}$$

$$= \frac{1}{BC(r)}\left(\frac{E^2}{mc^2}\frac{1}{A} - C^{1/2}(r)\left(mc^2 + \frac{1}{C(r)}\frac{L^2}{m}\right)\right) \quad -(8)$$

The equation of motion of Cartan's metric is therefore:

$$\boxed{\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2BC(r)}\left(\frac{E^2}{mc^2}\frac{1}{A} - C^{1/2}(r)\left(mc^2 + \frac{1}{C(r)}\frac{L^2}{m}\right)\right)}$$

$$-(9)$$

For spherical spacetime the same equation of motion is:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - n(r)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(10)$$

To obtain the orbital equation of the Cartan metric use:

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\frac{d\theta}{d\tau}, \quad \frac{d\theta}{d\tau} = \frac{L}{mC(r)}, \quad -(11)$$

so

$$\frac{dr}{d\tau} = \frac{L}{mC(r)}\frac{dr}{d\theta}, \quad m\left(\frac{dr}{d\tau}\right)^2 = \frac{L^2}{mC^2(r)}\left(\frac{dr}{d\theta}\right)^2$$

$$-(12)$$

Therefore:

$$\frac{L^2}{mC^2(r)}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{E^2}{mc^2 A} - C^{1/2}(r)\left(mc^2 + \frac{1}{C(r)}\frac{L^2}{m}\right)\right)\cdot\frac{1}{BC(r)}$$

$$-(13)$$

3)

i.e.

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{m\,C(r)}{BL^2}\left(\frac{E^2}{mc^2}\frac{1}{A} - C^{1/2}(r)\left(mc^2 + \frac{1}{C(r)}\frac{L^2}{m}\right)\right) \quad -(14)$$

The orbital equation is therefore:

$$\boxed{\frac{dr}{d\theta} = \frac{m^{1/2}}{L}\left(\frac{C(r)}{B}\right)^{1/2}\left(\frac{E^2}{mc^2}\frac{1}{A} - C^{1/2}(r)\left(mc^2 + \frac{1}{C(r)}\frac{L^2}{m}\right)\right)^{1/2}} \quad -(15)$$

For a spherical spacetime the orbital equation is:

$$\frac{dr}{d\theta} = r^2\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{1/2} \quad -(16)$$

where $a = \frac{L}{mc}$, $b = \frac{cL}{E}$. $-(17)$

In order to calculate the deflection of light due to gravitation from (15) the Crothers metric parameters E, L, B and $C(r)$ must be known. Therefore light deflection by gravitation cannot be calculated ab initio from any method based on the infinitesimal line element.