195(1): The Crothers Metric.

In spherical coordinates this is defined as:

$$ds^2 = A C^{1/2} c^2 dt^2 - B C^{1/2} dr^2 - C(d\theta^2 + \sin^2\theta\, d\phi^2) \quad -(1)$$

where A and B are constants, and where:

$$C = C(r) = (|r - r_0|^n + \alpha^n)^{2/n} \quad -(2)$$

The spherical coordinate system is defined by:

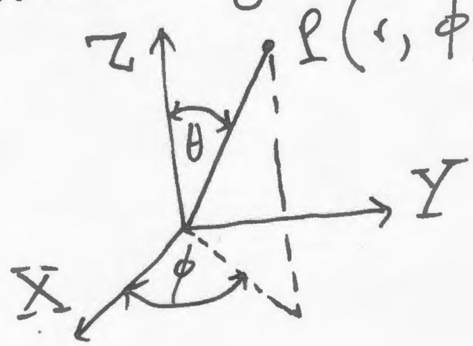

Fig (1)

and in Euclidean space:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \quad -(3)$$

The cylindrical coordinate system is defined by:

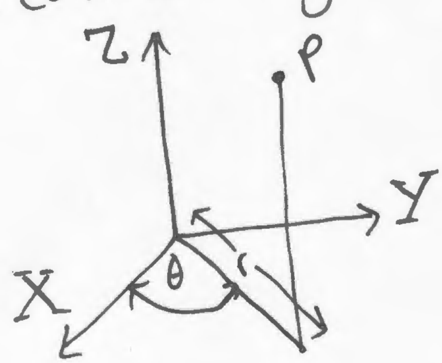

Fig. (2)

and in Euclidean space:

$$ds^2 = dr^2 + r^2 d\theta^2 + dZ^2 \quad -(4)$$

In the plane:

$$dZ^2 = 0 \quad -(5)$$

eq. (4) reduces to:

$$ds^2 = dr^2 + r^2 d\theta^2 \quad -(6)$$

2) In the spherical system:

$$\left.\begin{aligned} Z &= r\cos\theta \\ X &= r\sin\theta\cos\phi \\ Y &= r\sin\theta\sin\phi \end{aligned}\right] \quad -(7)$$

The infinitesimal line element in both systems is defined by:

$$ds^2 = d\underline{r} \cdot d\underline{r}, \quad -(8)$$

and is the same in both systems, so it follows that:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \quad -(9)$$
$$= dr^2 + r^2 d\theta^2 + dZ^2$$

i.e. with reference to Fig. (3):

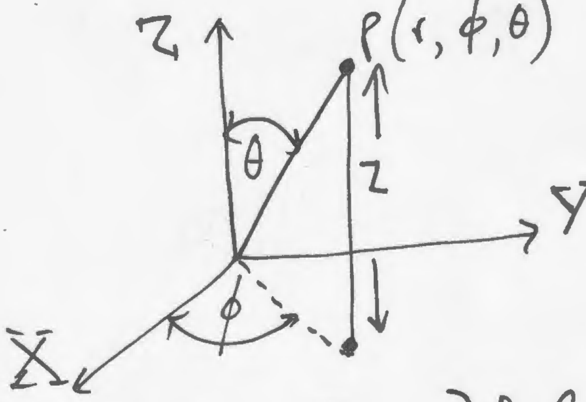

Fig. (3)

$$\left.\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \\ &= dr^2 + r^2 d\phi^2 + dZ^2 \end{aligned}\right\} -(10)$$

so the θ in Fig (2) is relabelled as ϕ. In the plane

$$dZ^2 = 0 \quad -(11)$$

$$ds^2 = dr^2 + r^2 d\phi^2 \quad -(12)$$

$$d\theta^2 = 0 \quad -(13)$$

i.e.

$$\sin^2\theta = 1 \quad -(14)$$

and

$$\theta = \pi \quad -(15)$$

Therefore the Cartan metric in the plane (5) is:

$$ds^2 = AC(r)^{1/2} c^2 dt^2 - BC(r)^{1/2} dr^2 - C(r) d\phi^2 \quad -(16)$$

and this can be relabelled as:

$$ds^2 = AC(r)^{1/2} c^2 dt^2 - BC(r)^{1/2} dr^2 - C(r) d\theta^2 \quad -(17)$$

$$= c^2 d\tau^2$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2} mc^2 = \frac{m}{2} \left(c^2 A C^{1/2}(r) \left(\frac{dt}{d\tau}\right)^2 - B C^{1/2}(r) \left(\frac{dr}{d\tau}\right)^2 - C(r) \left(\frac{d\theta}{d\tau}\right)^2 \right) \quad -(18)$$

The total energy is:

$$E = mc^2 A C^{1/2}(r) \frac{dt}{d\tau} \quad -(19)$$

and the total angular momentum is:

$$L = m\, C(r) \frac{d\theta}{d\tau}. \quad -(20)$$

In eq. (17):

$$d\underline{r} \cdot d\underline{r} = v^2 dt^2 = B C^{1/2}(r) dr^2 + C(r) d\theta^2 \quad -(21)$$

so

$$v^2 = B C^{1/2}(r) \left(\frac{dr}{dt}\right)^2 + C(r) \left(\frac{d\theta}{dt}\right)^2. \quad -(22)$$

It follows that:

$$c^2 d\tau^2 = \left(c^2 A C^{1/2}(r) - v^2\right) dt^2 \quad -(23)$$

$$\frac{dt}{} = \left(A C^{1/2}(r) - \frac{v^2}{c^2} \right)^{-1/2} \quad -(24)$$

4) From eqs. (19) and (24):

$$E = mc^2 A C^{1/2}(r)\left(A C^{1/2}(r) - \frac{v^2}{c^2}\right)^{-1/2} \quad -(25)$$

where:

$$v^2 = B C^{1/2}(r)\left(\frac{dr}{dt}\right)^2 + C(r)\left(\frac{d\theta}{dt}\right)^2 \quad -(26)$$

Eqs. (25) and (26) give a relation between A, B and $C(r)$.
