

235(4) : Some Rechs a Calculations and Concepts

It is first checked that:

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r \quad - (1)$$

using: $\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C}) \underline{B} - (\underline{A} \cdot \underline{B}) \underline{C}, \quad - (2)$

so: $\underline{\omega} \times (\underline{\omega} \times \underline{r}) = (\underline{\omega} \cdot \underline{r}) \underline{\omega} - (\underline{\omega} \cdot \underline{\omega}) \underline{r} \quad - (3)$
 $= -\omega^2 \underline{r} = -\omega^2 r \underline{e}_r$

QED. So in plane polar coordinates:

$$\underline{a} = \frac{d^2 \underline{r}}{dt^2} = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad - (4)$$

for any orbit. Eq. (4) means that the total acceleration \underline{a} is the second time derivative of the position vector \underline{r} in an observer frame. The rotation of the axes produces the additional acceleration

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r \quad - (5)$$

due to the spin connection $\underline{\omega}$.

In eq. (4), the part of the acceleration:

$$\underline{a}_N = \frac{d^2 r}{dt^2} \underline{e}_r = \underline{a} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad - (6)$$

2) is the Newtonian or inertial frame acceleration, i.e. the total acceleration \underline{a} minus the acceleration due to the rotation of the frame of reference

From previous work:

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \frac{dr}{d\theta} \frac{d}{dr} \left(\frac{1}{r^2} \frac{dr}{d\theta}\right) \quad - (7)$$

For the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos \theta} \quad - (8)$$

and

$$\frac{d^2 r}{dt^2} = \frac{L^2 \epsilon \cos \theta}{m^2 r^3 d}$$

i.e.

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{1}{r} - \frac{1}{d}\right) \quad - (9)$$

For a circular orbit:

$$r = d \quad - (10)$$

so

$$\frac{d^2 r}{dt^2} = 0 \quad - (11)$$

The force for eq. (9) is:

$$\underline{F}_N = \left(\frac{L^2}{mr^3} - \frac{L^2}{mr^2 d} \right) \underline{e}_r \quad - (12)$$

in the inertial frame i.e. which there is no spin
correction.

3) In Q Newtonian theory :

$$d = \frac{L^2}{m^2 M G} \quad - (13)$$

and

$$\underline{F}_N = \left(\frac{L^2}{mr^3} - \frac{mMG}{r^2} \right) \underline{e}_r \quad - (14)$$

and this is interpreted as the force of attraction with the minus sign being counterbalanced by a "pseudo-force":

$$\underline{F}_C = - \frac{2Mc}{2r} \underline{e}_r = \frac{L^2}{mr^3} \underline{e}_r.$$

However, the complete acceleration is, $- (15)$

from eq. (4):

$$\underline{a} = \left(\frac{L^2}{mr^3} - \frac{L^2}{mr^3 d} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \right) \underline{e}_r \quad - (16)$$

where

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -\omega^2 r \underline{e}_r \quad - (17)$$

$$= -\frac{L^2}{mr^3} \underline{e}_r$$

so

$$\underline{a} = -\frac{L^2}{mr^3 d} \underline{e}_r \quad - (18)$$

which is the acceleration produced by an

4) elliptical orbit:

$$r = \frac{d}{1 + e \cos \theta} \quad - (19)$$

Eq. (19) means that:

$$\underline{r} = \left(\frac{d}{1 + e \cos \theta} \right) \underline{e}_r \quad - (20)$$

and

$$\underline{a} = \frac{d^2 \underline{r}}{dt^2} = - \frac{L^2}{m r^3 d} \underline{e}_r \quad - (21)$$

where

$$L = m r^2 \omega \quad - (22)$$

So:

$$\underline{a} = - \left(\frac{r^2}{d} \right) \omega^2 \underline{e}_r \quad - (23)$$

i.e.

$$\underline{a} = \frac{r}{d} \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad - (24)$$

if $\underline{r} = \left(\frac{d}{1 + e \cos \theta} \right) \underline{e}_r \quad - (25)$

In the case of a circular orbit:

$$\underline{a} = \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad - (26)$$

$$= - \omega^2 r \underline{e}_r$$

this is a purely kinematic result:

5)
$$\frac{d^2 \underline{r}}{dt^2} = -\omega^2 \underline{r} \quad (27)$$

for a circle, where:

$$\underline{r} = r \underline{e}_r, \quad (28)$$

so
$$\boxed{\frac{d^2 \underline{r}}{dt^2} = -\omega^2 \underline{r}} \quad (29)$$

For ellipse:
$$\boxed{\frac{d^2 \underline{r}}{dt^2} = -\left(\frac{r}{a}\right) \omega^2 \underline{r}} \quad (30)$$

A solution of eq. (29) is:

$$\underline{r} = \underline{r}(0) \exp(i\omega t) \quad (31)$$

and
$$\text{Real}(\underline{r}) = \underline{r}(0) \cos(\omega t) \quad (32)$$

so the vector \underline{r} rotates in a circle with angular velocity ω . This is also the spin connection,

so the dynamics are those of free particle.

In the Newtonian theory the vector \underline{r} does not rotate, and so it is an incomplete theory.