

235(e) : Cartan Curvature of the Plane Polar Coordinates

The Cartan curvature is defined by:

$$R^a_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c} \omega^c_{\nu b} - \omega^a_{\nu c} \omega^c_{\mu b} \quad - (1)$$

For the plane polar coordinates (r, θ) :

$$\omega^{(1)}_{0(2)} = -\omega^{(2)}_{0(1)} = \omega \quad - (2)$$

$$\omega^{(1)}_{0(0)} = \omega^{(1)}_{0(1)} = \omega^{(2)}_{0(2)} = \omega^{(2)}_{0(0)} = 0 \quad - (3)$$

and

$$\begin{bmatrix} \omega^{(1)}_{1(0)} & \omega^{(1)}_{2(0)} \\ \omega^{(2)}_{1(0)} & \omega^{(2)}_{2(0)} \end{bmatrix} = \omega \begin{bmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} \quad - (4)$$

Therefore:

$$\begin{aligned} R^{(1)}_{(2)01} &= \partial_0 \omega^{(1)}_{1(2)} - \partial_1 \omega^{(1)}_{0(2)} \\ &\quad + \omega^{(1)}_{0(0)} \omega^{(0)}_{1(2)} - \omega^{(1)}_{1(0)} \omega^{(0)}_{0(2)} \\ &= -\partial_1 \omega^{(1)}_{0(2)} - \omega^{(1)}_{1(0)} \omega^{(0)}_{0(2)} \\ &= -2 \frac{d\omega}{dr} \quad - (5) \end{aligned}$$

by antisymmetry, i.e.:

$$\partial_1 \omega^{(1)}_{0(2)} = -\omega^{(1)}_{1(0)} \omega^{(0)}_{0(2)} \quad - (6)$$

2) Similarly:

$$R^{(2)}_{(1)01} = -\partial_1 \omega^{(2)}_{0(1)} - \omega^{(2)}_{1(0)} \omega^{(0)}_{0(1)} \\ = -2 \frac{d\omega}{dr} \quad - (7)$$

$$R^{(1)}_{(2)02} = -\partial_2 \omega^{(1)}_{0(2)} - \omega^{(1)}_{2(0)} \omega^{(0)}_{0(2)} \\ = -2 \frac{d\omega}{d\theta} \quad - (8)$$

$$R^{(2)}_{(1)02} = -\partial_2 \omega^{(2)}_{0(1)} - \omega^{(2)}_{2(0)} \omega^{(0)}_{0(1)} \\ = -2 \frac{d\omega}{d\theta} \quad - (9)$$

Summary

$$R^{(1)}_{(2)01} = -R^{(2)}_{(1)01} = -2 \frac{d\omega}{dr} \quad - (10)$$

$$R^{(1)}_{(2)02} = -R^{(2)}_{(1)02} = -2 \frac{d\omega}{d\theta}$$

The four tensor elements are given by:

$$\begin{bmatrix} T^{(1)}_{01} & T^{(1)}_{02} \\ T^{(2)}_{01} & T^{(2)}_{02} \end{bmatrix} = 2\omega \begin{bmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \quad - (11)$$

They are related by Cartan identity:

3)

$$DNT := R \Lambda \dot{q} - (12)$$

which leads to field equations.

In Newtonian theory ϕ orbit is an ellipse or conical section:

$$r = \frac{d}{1 + \epsilon \cos \theta} - (13)$$

so
$$\cos \theta = \frac{1}{\epsilon} \left(\frac{d}{r} - 1 \right) - (14)$$

$$\begin{aligned} \sin \theta &= (1 - \cos^2 \theta)^{1/2} \\ &= \left(1 - \frac{1}{\epsilon^2} \left(\frac{d}{r} - 1 \right)^2 \right)^{1/2} - (15) \end{aligned}$$

By conservation of angular momentum L :

$$\omega = \frac{L}{mr^2} - (16)$$

so
$$\frac{d\omega}{dr} = - \frac{2L}{mr^3} - (17)$$

and
$$\frac{d\omega}{d\theta} = \frac{d\omega}{dr} \frac{dr}{d\theta} - (18)$$

with
$$\frac{dr}{d\theta} = \frac{\epsilon r^2 \sin \theta}{d} - (19)$$

so
$$\frac{d\omega}{d\theta} = - \frac{2L \epsilon r^2 \sin \theta}{d m r^3} - (20)$$

4) i.e. $\frac{d\omega}{dt} = -\frac{2\epsilon L}{dmr} \sin\theta \quad - (21)$

Therefore:

$$R^{(1)}_{(2)01} = -R^{(2)}_{(1)01} = \frac{4L}{mr^3} \quad - (22)$$

$$R^{(1)}_{(2)02} = -R^{(1)}_{(2)02} = \frac{4\epsilon L}{dmr} \sin\theta \quad - (23)$$

and $T^{(1)}_{02} = -T^{(2)}_{01} = \frac{2L}{mr^2\epsilon} \left(\frac{d}{r} - 1 \right) \quad - (24)$

In Newtonian theory the centrifugal pseudo force is $F_c = \frac{L^2}{mr^3} \quad - (25)$

and it is seen that:

$$F_c = \frac{LR^{(1)}_{(2)01}}{4} \quad - (26)$$

i.e. the centrifugal pseudo force is due to
on element of the Cartan curvature.