11

Derivation Of The Geometrical Phase From The Evans Phase Law Of Generally Covariant Unified Field Thoery

Summary. The phase law of generally covariant electrodynamics is used to explain straightforwardly the origin of the geometrical and Berry phase effects, exemplified by the Tomita-Chiao effect. Both effects are described by a phase factor that is constructed from the generally covariant Stokes formula of differential geometry, a phase factor in which the contour integral over the potential field $\boldsymbol{A}^{(3)}$ is equated to the area integral over the gauge invariant field $\boldsymbol{B}^{(3)}$, the Evans-Vigier field. The latter is the fundamental spin Casimir invariant of the Einstein group of general relativity applied to electrodynamics. General relativity as extended in the Evans unified field theory is needed for a correct understanding of all phase effects in physics, an understanding that is forged through the Evans phase law, the origin both of the Berry phase and the geometrical phase of electrodynamics observed in the Sagnac and Tomita-Chiao effects.

Key words: geometrical phase, Tomita-Chiao effect, Berry phase, Evans-Vigier field, generally covariant electrodynamics, $\boldsymbol{B}^{(3)}$ field.

11.1 Introduction

Phase factors such as the Berry phase or that observed in the Tomita-Chiao effect are due in general [1,2] to parallel transport in the presence of a gauge field. This inference suggests that such phase factors should properly be described by covariant derivatives in general relativity. Recently a unified field theory has been developed in which electrodynamics is generally covariant [3-9], i.e., becomes understandable with general relativity as required by the Einsteinian principle that all theories of natural philosophy be theories of general relativity. One of the outcomes of this theory is the generally covariant Evans phase law of the unified field, a phase factor which is constructed [10,11] from the generally covariant Stokes formula of differential geometry [12,13]. The field theory is made generally covariant by replacing the exterior derivative of differential geometry, denoted $d\wedge$, by the covariant exterior derivative, denoted by $D\wedge$. Therefore the magnetic field, for example, is defined by the first Maurer Cartan structure relation [12]

$$B = D \wedge A = d \wedge A + gA \wedge A, \tag{11.1}$$

where A is the potential field and g is a proportionality factor with the units of inverse magnetic flux $(e/\hbar)$. In the older Maxwell Heaviside (MH) field theory the magnetic field is defined by

$$B = d \wedge A, \tag{11.2}$$

and it is therefore not a generally covariant field theory. For electrodynamics the Evans phase law is

$$\Phi = \exp\left(ig \oint \boldsymbol{A}^{(3)} \cdot d\boldsymbol{r}\right) = \exp\left(ig \oint \boldsymbol{B}^{(3)} \cdot \boldsymbol{k} dAr\right) := \exp(i\Phi_E), \tag{11.3}$$

where $\boldsymbol{A}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are directed in the propagation axis of the electromagnetic beam and where Ar is the area enclosed by the beam. The Z axis in Eq. (11.3) is the propagation axis of the beam. For matter fields the phase law (11.3) becomes

$$\Phi = \exp\left(i \oint \boldsymbol{\kappa} \cdot d\boldsymbol{r}\right) = \exp\left(i \int \boldsymbol{\kappa}^2 dAr\right) := \exp(i\Phi_E), \tag{11.4}$$

where κ is the wave number [10]. The phase law (11.3) and (11.4) is the first correct phase law of field theory, and gives the first correct explanation [10] of well known phenomena of physical optics such as reflection, interferometry, the Sagnac and Aharonov Bohm (AB) effects. The MH field theory is unable to describe these effects because it is an incomplete theory [2-10] of special relativity.

In Sec. 2 of the Tomita-Chiao effect [1,2] is derived from the Evans phase law (11.4), and in Sec. 3 the Berry phase of matter fields [1,2] is derived from the equivalent phase law (11.4). It is concluded that the origin of the Berry phase is general relativity as developed in the Evans unified field theory [3-10].

11.2 Derivation Of The Tomita-Chiao Effect From The Evans Phase Law

The phase law (11.4) results in a rotation of plane polarized radiation (50% right- and 50% left-circularly polarized) upon propagation in a helical path:

$$\boldsymbol{I}_e = (\boldsymbol{i} - i\boldsymbol{j})(e^{i\phi} + e^{-i\phi}) = 2\boldsymbol{i}\cos\phi. \tag{11.5}$$

To see this, consider initially plane-polarized light, defined as a sum of 50% right- and 50% left-circularly polarized radiation:

$$\begin{aligned} \boldsymbol{I}_L &= \mathrm{Re}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi} = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}, \\ \boldsymbol{I}_R &= \mathrm{Re}(\boldsymbol{i} - i\boldsymbol{j})e^{-i\phi} = \cos\phi\boldsymbol{i} - \sin\phi\boldsymbol{j}. \end{aligned} \tag{11.6}$$

After light has propagated along the arc length s of a helix, the phase factor in Eq. (11.5) becomes

$$\boldsymbol{I}'_e = (\boldsymbol{i} - i\boldsymbol{j})e^{i(\phi+\Phi_E)} + (\boldsymbol{i} - i\boldsymbol{j})e^{-i(\phi-\Phi_E)} = \exp(i\Phi_E)\boldsymbol{I}_e \tag{11.7}$$

as a result of the Evans phase law. The angle Φ_E is the Evans phase. Equation (11.7) is generally covariant, and an example of the principle of least curvature [9]. The Tomita-Chiao effect [1,2] is therefore the observation of the Berry phase [14] by rotation of the plane of linearly polarized light due to the scalar curvature R of the helical optical fibre through which the light propagates. After propagation over a distance s along the helix, the polarization changes to

$$\boldsymbol{I}'_e = 2\cos\phi\left(\cos\Phi_e\boldsymbol{i} - \sin\Phi_e\boldsymbol{j}\right), \tag{11.8}$$

where we have used the angle formulas

$$\begin{aligned} \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B, \\ \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B. \end{aligned} \tag{11.9}$$

Initially the plane polarized light was polarized as

$$\boldsymbol{I}_e = 2\cos\phi\boldsymbol{i}; \tag{11.10}$$

so, after a propagation distance Z, the plane of polarization has changed from (11.5) to (11.8) due to the Evans phase law (11.4). In MH electrodynamics there is no such effect because the spacetime of the theory is flat, so $R = 0$. In MH theory the phase is purely dynamical, and given by

$$I_{MH} = \exp\left(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r})\right), \tag{11.11}$$

so there is no mechanism available in special relativity with which to change the plane of polarization of light propagating through a helical optical fibre. Therefore, the Tomita-Chiao effect proves experimentally that the spacetime of electrodynamics is non-Euclidean, with non-zero scalar curvature R. The angle through which the plane of the light is rotated is given from Eq. (11.8) as

$$\tan\theta = \frac{\sin\phi_S}{\cos\phi_S}, \tag{11.12}$$

from which it is inferred that

$$\theta = \Phi_E. \tag{11.13}$$

Therefore the angle through which the plane of light is rotated in the Tomita-Chiao effect, or in any Berry phase, originates in the Evans phase (11.4) of unified field theory and is given in general by

$$\theta = \kappa \oint ds = R \int dAr. \tag{11.14}$$

In this equation, R is the scalar curvature of a given spacetime, or base manifold, and κ is a wavenumber (inverse wavelength) associated with the wave nature of the spacetime or base manifold. Thus matter waves and electromagnetic wave are manifestations of spacetime itself as required in general relativity. In Eq. (11.14), ds is an infinitesimal line element of the non-Euclidean spacetime. Significantly, Einstein's original theory of general relativity originated from the fact that the square ds^2 of the line element is an invariant of general coordinate transformation [12]. The helix defines a rotating and translating baseline in a non-Euclidean manifold. Similarly, the rotating and translating transverse electromagnetic potential vector, defined by

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \tag{11.15}$$

describes the helical baseline of non-Euclidean geometry in the Evans unified field theory [3-10]. The vector $\boldsymbol{A}^{(1)}$ is the spacetime geometry itself multiplied by $A^{(0)}$. The helix in the Tomita-Chiao effect is a physical object that guides $A^{(0)}$ along a helical path, creating a base manifold, or spacetime, with a helical baseline. So the Evans phase law (11.14) applies equally well to both situations. Similarly, the Evans phase law describes straightforwardly [10] the Sagnac and AB effects. The latter is a type of Berry phase [1,2,15].

In the original Tomita-Chiao effect [1,2[it was observed experimentally that the plane of polarization of light is changed after propagation through a helical optical fiber. If the helix is parameterized [10,16] by

$$x = x_0 \cos\theta, \quad y = y_0 \sin\theta, \quad z = z_0\theta, \tag{11.16}$$

then

$$\frac{dx}{d\theta} = -x_0 \sin\theta, \quad \frac{dy}{d\theta} = y_0 \cos\theta, \quad \frac{dz}{d\theta} = z_0. \tag{11.17}$$

The wavenumber vector in general is

$$\boldsymbol{\kappa} = \kappa_x \boldsymbol{i} + \kappa_y \boldsymbol{j} + \kappa_z \boldsymbol{k}, \tag{11.18}$$

and so the contour integral appearing on the right-hand side of Eq. (11.14) is

$$\oint \boldsymbol{k} \cdot d\boldsymbol{r} = -\kappa_x x_0 \oint \sin\theta d\theta + \kappa_y y_0 \oint \cos\theta d\theta + \kappa_z z_0 \oint \theta d\theta. \tag{11.19}$$

Integration from 0 to 2π of Eq. (11.19) produces the result

$$\kappa \oint ds = \oint \phi \boldsymbol{\kappa} \cdot d\boldsymbol{r} = z_0 \int_0^{2\pi} \kappa_z \theta d\theta = 2\pi^2 \kappa_z z_0 \tag{11.20}$$

because

$$\int_0^{2\pi} \sin\theta d\theta = \int_0^{2\pi} \cos\theta d\theta = 0. \tag{11.21}$$

Therefore the angle of rotation of plane polarized light in the Tomita-Chiao effect is given by

$$\theta = 2\pi^2 \kappa_z z_0 = \kappa_z^2 Ar, \tag{11.22}$$

where Ar is the area of a circle whose circumference $2\pi r$ is equal to the arc length of the helical fibre. (A helix can always be constructed from a circle by cutting the circle, pulling it out into a line, and winding the line on a cylinder. The arc length of the helix so constructed must be the same as the circumference of the original circle. If the circle is pulled out into a straight line, the length of the line is the circumference of the original circle.) By integrating from 0 to 2π in Eq. (11.19), we have considered a special case for simplicity of argument.

Equation (11.22) is therefore the result of the general formula (11.14) applied to a helix of arc length $s = 2\pi r$. On writing $\kappa_z = z_1^{-1}$, the Tomita-Chiao phase can finally be expressed as

$$\theta = 2\pi^2 z_0 / z_1. \tag{11.23}$$

The Tomita-Chiao phase is conventionally described [1,2] by

$$\begin{aligned} \exp(-i\theta) &= \exp\left(-i2\pi(1 - \cos\lambda)\right) \\ &= \exp\left(-2\pi\right) i \exp(2\pi i \cos\lambda) \\ &= \exp(2\pi i \cos\lambda), \end{aligned} \tag{11.24}$$

and this is the same as Eq. (11.23) upon identifying

$$\cos\lambda = \pi z_0 / z_1. \tag{11.25}$$

Therefore we have derived the experimentally observed Tomita-Chiao phase from the Evans phase law of generally covariant unified field theory [10]. In so doing the Tomita-Chiao phase is recognized as a phenomenon of general relativity in which spacetime itself has a given non-Euclidean geometry, in this case helical in nature.

11.3 Derivation Of The Berry Phase From The Evans Phase

The Tomita-Chiao effect [1,2] is considered to be the first experimental observation of the Berry phase [14], and is sometimes known as the optical Berry phase or Hannay angle [17]. Such a phase shift occurs whenever a physical phenomenon is defined by a closed path in state space or parameter space [15].

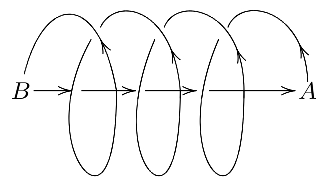

Fig. 11.1.

The closed path in the Evans phase law is defined by the generally covariant Stokes law [3-10] of differential geometry, in which the exterior derivative $d\wedge$ is replaced wherever it occurs with the covariant exterior derivative $D\wedge$. Thus, a magnetic field of any type is always defined in generally covariant electrodynamics in a non-Euclidean spacetime by

$$B = D \wedge A \tag{11.26}$$

and not by

$$B = d \wedge A, \tag{11.27}$$

as in MH field theory in flat (i.e., Euclidean) spacetime. This realization gives rise [18] to the Evans-Vigier field $\boldsymbol{B}^{(3)}$ and to $O(3)$ electrodynamics [19], an example of generally covariant electrodynamics [3-10] in which the orthonormal space of differential geometry (indexed a [12]) is described by the complex circular basis (1),(2), (3)) with $O(3)$ symmetry. The complex circular basis is a natural description of circular polarization [18].

The closed path for the helix is defined as follows, from A to B along the helix and back from B to A along the axis of the helix; see Fig. (11.1). In terms of topology [13] the Evans phase (and therefrom the Berry phase) originates in the fact that a helix cannot be shrunk to a point, and so is not simply connected. A circle can be shrunk to a point, a helix shrinks to a line, the axis of the helix along Z. This procedure is analogous to drawing a helix out into a straight line along Z. Therefore a covariant Stokes theorem must be used to describe the closed path in Fig. (11.1), because the path back from B to A along the axis Z of the helix cuts through the center of the path from A to B along the helix. The generally covariant and non-Abelian Stokes theorem needed to describe this path is [10]

$$\oint \boldsymbol{\kappa}^{(3)} \cdot d\boldsymbol{r} = -i \int \boldsymbol{\kappa}^{(1)} \times \boldsymbol{\kappa}^{(2)} \cdot \boldsymbol{k} dAr, \tag{11.28}$$

and this is the phase of the Evans phase law (11.14). It has been shown elsewhere [3-10] that Eq. (11.28) quantitatively describes many phenomena which the MH phase (11.11) cannot describe qualitatively. In dynamics, the Foucault pendulum is another example of the Berry phase [15], now known to originate in the Evans phase of generally covariant unified field theory [3-10], and in electrodynamics the Pancharatnam phase [20] has similar characteristics and origin.

As discussed in reference [15] any vector parallel transported in a closed path produces the Berry phase. This procedure in the Evans unified field theory is an outcome of general relativity as argued already. As described in Ref. [15], the rotation angle of the Berry phase is related to the integral of the curvature of the surface bounded by the loops. This is precisely what is shown by the Evans phase in the forms (11.4), (11.14), or (11.28). The curvature of the helix is

$$R = \kappa^2, \tag{11.29}$$

and this realization gives rise [10,18] to the Sagnac effect in light traversing a circular path, a circle whose circumference is the same as the arc length of the helix drawn out from the circle as argued already.

So the Tomita-Chiao and Sagnac effects are essentially the same [18], and both are geometrical/topological phases originating in the Evans phase of general relativity, as does the Berry phase and Pancharatnam phase and all effects in physics in which these phases are observed [15]. All these effects therefore serve as further experimental verification of the Evans generally covariant unified field theory [3-10]. The Evans phase law is more fundamental than the Berry phase law because the former self-consistently describes both the dynamical phase of physical optics (observed for example in ordinarily reflection and interferometry) is an example of the Evans phase where the wave number κ in Eq. (11.14) is a property of the radiation itself (the electromagnetic field). The Berry phase is an example of the Evans phase where the wavenumber is an inverse distance or inverse periodic length or wavelength defined by the wavelength (or scalar curvature R) of spacetime itself as required by general relativity. In the Tomita-Chiao effect the set-up is an optical fiber wound into a helix. Similarly, the area in Eq. (11.14) is either a property of the radiation (dynamical phase from Evans phase) or of the set up (geometrical/topological phase from Evans phase). The Pancharatnam phase [20] can similarly be derived from the Evans phase by considering a closed loop in spacetime in given optical configurations. The Foucault pendulum is similarly an outcome of the Evans phase law applied to dynamics, as is the Sagnac and AB effects, and indeed, all effects in optics. The dynamical phase in optics always measures the $\boldsymbol{B}^{(3)}$, or Evans-Vigier, field [18,19].

The Evans phase law is intrinsically gauge invariant [10] because it always uses covariant derivatives. Self-consistently, the Berry phase is also gauge invariant for the same reason: It is a geometrical phase [15]. States in quantum mechanics are wavefunctions, and also acquire a Berry phase in general [15]. This well-known result is now understandable through the Evans phase and principle of least curvature [9], which unifies the Hamilton principle of least action and the Fermat principle of least time, giving rise to wave mechanics [9] and to the Schrödinger equation in the quantum weak field limit of the Evans unified field theory. The Berry phase in quantum mechanics becomes part of the Evans phase, a part that is observable experimentally in given configurations [1,2].

These inferences illustrate the fact that the generally covariant unified field theory [3-10] is a powerful and general theory of radiated and matter fields, and of geometrical and phases and related effects.

Acknowledgments

The Ted Annis Foundation, Craddock Inc., and ADAS are thanked for funding, and members of staff of AIAS and others for many interesting discussions.

References

1. A Tomita and R. Y. Chiao, *Phys. Rev. Lett.* **57**, 937 (1986).
2. T. W. Barrett in A. Lakhtakia, ed., *Essays on the Formal Aspects of Electromagnetic Theory* (World Scientific, Singapore, 1992).
3. M. W. Evans, *Found. Phys. Lett.* **16**, 367 (2003).
4. M. W. Evans, *Found. Phys. Lett.* **16**, 507 (2003).
5. M. W. Evans, *Found. Phys. Lett.* **17**, 25 (2004).
6. M. W. Evans, "Derivation of Dirac's equation from the Evans wave equation," *Found. Phys. Lett.* **17**(2) (2004), in press; preprint on www.aias.us.
7. M. W. Evans, "Unification of the gravitational and strong nuclear fields," *Found. Phys. Lett.* (2004), in press; preprint on www.aias.us
8. M. W. Evans, "The Evans lemma of differential geometry," *Found. Phys. Lett.*, submitted for publication; preprint on www.aias.us; "Derivation of the Evans wave equation from the Lagrangian and action, origin of the Planck constant in generally relativity," *Found. Phys. Lett.*, in press; preprint on www.aias.us; "Development of the Evans wave equation in the weak-field limit, the electrogravitic equation," *Found. Phys. Lett.*, in press; preprint on www.aias.us.
9. L. Felker, ed., *The Evans Equation* (World Scientific, Singapore, 2004, in preparation).
10. M. W. Evans, "Physical optics, the Sagnac effect and the Aharonov-Bohm effect in the Evans unified field theory," *Found. Phys. Lett.*, in press; preprint on www.aias.us.
11. M. W. Evans, "Generally covariant field and wave equations for gravitation and quantized gravitation in terms of the metric four vector, *Found. Phys. Lett.*, submitted for publication, preprint on www.aias.us.
12. S. M. Carroll, *Lecture Notes on General Relativity* (University of California, Santa Barbara, graduate course), arXiv:gr-qe/9712019 vl 3 Dec., 1997.
13. L. H. Ryder, *Quantum Field Theory*, 2nd edn. (University Press, Cambridge, 1996).
14. M. V. Berry, *Proc. Roy. Soc. A* **392**, 54 (1984).
15. www.mi.infn.it/manini/berryphase.
16. E. M. Milewski, ed., *Vector Analysis Problem Solver* (Research and Education Association, New York, 1987).
17. J. H. Hannay, *J. Phys. A* **18**, 221 (1985).

18. M. W. Evans, "O(3) electrodynamics," in M. W. Evans, ed., *Modern Nonlinear Optics*, a special topical issue of I. Prigogine and S. A. Rice, series eds., *Advances in Chemical Physics*, Vol. 119 (2), 2nd and e-book editions (Wiley Interscience, New York, 2001).
19. M. W. Evans, J.-P. Vigier, *et al.*, *The Enigmatic Photon* (Kluwer/Academic, Dordrecht, 1994 to 2003, hardback and paperback) in 10 volumes. M. W. Evans and L. B. Crowell, *Classical and Quantum Electrodynamics and the* $\boldsymbol{B}^{(3)}$ *Field* (World Scientific, Singapore, 2001).
20. T. W. Barrett and D. M. Grimes, *Advanced Electromagnetism* (World Scientific, Singapore, 1996).