

The Incompatibility of the Christoffel Connection with the Bianchi Identity

by

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Abstract

It is shown that the use of the Christoffel connection in general relativity is inconsistent with the Bianchi identity of differential geometry. This finding means that the Einstein Hilbert (EH) theory is not a valid theory of physics because it is based on the Ricci cyclic equation in which the Cartan torsion is missing. When the Cartan torsion is correctly re-instated in the Bianchi identity, it is found that the Christoffel connection is not a valid solution in the theory of gravitation. More generally, Einstein Cartan Evans (ECE) theory is needed for a self consistent development of the gravitational field unified with other fundamental fields such as electromagnetism.

Keywords: Incompatibility of the Christoffel connection and Bianchi identity; Einstein Hilbert field theory, Einstein Cartan Evans (ECE) field theory.

8.1 Introduction

Recently a relatively straightforward and generally covariant unified field theory has been suggested [1–10] based on the well known Cartan geometry [11]. This is known as Einstein Cartan Evans (ECE) field theory because it is based directly on the well known correspondence between Einstein and Cartan in the early part of the twentieth century. It is internationally accepted [12] that ECE is a valid and well tested field theory which has essentially supplanted the 1915 Einstein Hilbert (EH) field theory. In Section 8.2 one of the basic

weaknesses of EH theory is proven using the well known [11] Bianchi identity of differential geometry. This was developed by Cartan and is a cyclic sum of definitions of the Riemann tensor for any connection. The Riemann and torsion tensors are in turn derived from the commutator of covariant derivatives acting on the general four vector [1–11]. Both tensors are well defined for any connection irrespective of metric compatibility. One tensor cannot exist without the other. The basic weakness of the EH theory is the complete neglect of Cartan torsion. The so-called first Bianchi identity used in EH theory is in fact the Ricci cyclic equation, in which torsion does not appear at all. The correct Bianchi identity of differential geometry must include a non-zero torsion. In Section 8.2 this is developed in tensor notation and it is shown that the Christoffel connection cannot be a valid solution of the Bianchi identity. There is only one Bianchi identity [1–11], the so called first and second Bianchi identities of EH theory are approximations that neglect the Cartan torsion. In Section three a method is illustrated in which the interaction of two fields is considered in ECE theory, and an equation developed in which this basic weakness of EH theory is approximately circumvented.

8.2 Development of the Bianchi Identity and its Incompatibility with the Christoffel Connection

The Bianchi identity of Cartan’s standard differential geometry [11] is expressed most succinctly and in indexless notation [1–10] as:

$$D \wedge T := R \wedge q. \quad (8.1)$$

It is seen that it is the balance of torsion (T) and curvature (R). Here $D \wedge$ is a well defined covariant exterior derivative, T is Cartan’s torsion form, R is Cartan’s curvature form, and q is Cartan’s tetrad. The condensed notation (8.1) reveals the basic structure of the identity the most clearly. The symbol $:=$ denotes that it is an exact identity. One side is identically equal to the other side. Restoring the indices of Cartan’s tangent space-time [1–11], Eq. (8.1) becomes:

$$d \wedge T^a + \omega^a_b \wedge T^b := R^a_b \wedge q^b \quad (8.2)$$

where the spin connection ω^a_b has been written out in full. Translating to tensor notation Eq. (8.2) becomes [1–11]:

$$\begin{aligned} & \partial_\mu T^a_{\nu\rho} + \partial_\rho T^a_{\mu\nu} + \partial_\nu T^a_{\rho\mu} + \omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu} \\ & := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \end{aligned} \quad (8.3)$$

using the well known rules [1–11] for the wedge product. Now define the Hodge dual tensors [1–11]:

$$\tilde{R}^a{}_{\rho}{}^{\alpha\beta} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\alpha\beta\mu\nu} R^a{}_{\rho\mu\nu} \quad (8.4)$$

and

$$\tilde{T}^{a\alpha\beta} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\alpha\beta\mu\nu} T^a{}_{\mu\nu} \quad (8.5)$$

in four dimensions. The rule for Hodge duals [1–11] means that the square root of the determinant of the metric $g_{\mu\nu}$ must premultiply the 4-D Levi-Civita symbol of the Hodge dual in a base manifold with curvature and torsion. If metric compatibility is assumed then:

$$D_{\rho} g_{\mu\nu} = D_{\rho} g^{\mu\nu} = 0 \quad (8.6)$$

and the cyclic sums in Eq. (8.3) translate into [1–11]:

$$D_{\mu} \tilde{T}^{a\mu\nu} := \tilde{R}^a{}_{\mu}{}^{\mu\nu} \quad (8.7)$$

This is the tensorial version of the condensed identity (8.1). In the tensorial version appear the Hodge duals of the Cartan torsion tensor and the curvature tensor $\tilde{R}^a{}_{\mu}{}^{\mu\nu}$. A particular solution of Eq. (8.7) in the base manifold is [1–11]

$$D_{\mu} \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu}. \quad (8.8)$$

Eq. (8.1) may also be written [1–11] as:

$$\begin{aligned} R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\mu\nu\rho} + R^{\lambda}{}_{\nu\rho\mu} &:= \partial_{\mu} \Gamma^{\lambda}_{\nu\rho} - \partial_{\nu} \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\sigma}_{\mu\rho} \\ &\quad + \partial_{\nu} \Gamma^{\lambda}_{\rho\mu} - \partial_{\rho} \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\sigma}_{\nu\sigma} \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\lambda}_{\rho\sigma} \Gamma^{\sigma}_{\nu\mu} \\ &\quad + \partial_{\rho} \Gamma^{\lambda}_{\mu\nu} - \partial_{\mu} \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\rho\sigma} \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\rho\nu} \end{aligned} \quad (8.9)$$

i.e. as a cyclic sum of definitions of the Riemann tensor:

$$\begin{aligned} R^{\lambda}{}_{\rho\mu\nu} &:= \partial_{\mu} \Gamma^{\lambda}_{\nu\rho} - \partial_{\nu} \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\sigma}_{\mu\rho}, \\ R^{\lambda}{}_{\mu\nu\rho} &:= \partial_{\nu} \Gamma^{\lambda}_{\rho\mu} - \partial_{\rho} \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\lambda}_{\rho\sigma} \Gamma^{\sigma}_{\nu\mu}, \\ R^{\lambda}{}_{\nu\rho\mu} &:= \partial_{\rho} \Gamma^{\lambda}_{\mu\nu} - \partial_{\mu} \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\rho\sigma} \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\rho\nu} \end{aligned} \quad (8.10)$$

for any connection. The Riemann and torsion tensors are furthermore related by the basic equation [1–11]:

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}D_\lambda V^\rho. \quad (8.11)$$

One tensor cannot exist without the other and Eq. (8.11) shows that they are always defined by:

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu\Gamma^\lambda{}_{\nu\rho} - \partial_\nu\Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho}, \quad (8.12)$$

and

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \quad (8.13)$$

In general:

$$R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\nu\rho\mu} \neq 0. \quad (8.14)$$

Eq. (8.9) shows that Eq (8.1) is indeed an exact identity. One side is a restatement of the other side by definition. Eq. (8.11) shows that both tensors must be anti-symmetric in their last two indices:

$$R^\lambda{}_{\rho\mu\nu} = -R^\lambda{}_{\rho\nu\mu}, \quad (8.15)$$

$$T^\lambda{}_{\mu\nu} = -T^\lambda{}_{\nu\mu}, \quad (8.16)$$

and so have well defined Hodge duals in four dimensions [1–11]:

$$\tilde{R}^\lambda{}_{\rho\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu\alpha\beta}R^\lambda{}_{\rho}{}^{\alpha\beta}, \quad (8.17)$$

$$\tilde{T}^\lambda{}_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu\alpha\beta}T^{\lambda\alpha\beta}. \quad (8.18)$$

The Hodge dual of the torsion tensor is the same as the Hodge dual of the difference of anti-symmetric connections, so:

$$\tilde{T}^\lambda{}_{\mu\nu} = (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu})_{HD}. \quad (8.19)$$

There also exists a Hodge dual of the commutator of covariant derivatives, because this is also an anti-symmetric operator:

$$[D_\mu, D_\nu]_{HD} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu\alpha\beta}[D^\alpha, D^\beta]. \quad (8.20)$$

So we obtain the Hodge dual of Eq. (8.11):

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda_{\mu\nu} D_\lambda V^\rho \quad (8.21)$$

where the Hodge dual curvature and torsion tensors are defined by the Hodge dual commutator acting on the arbitrary four vector V^ρ .

In general the Hodge dual curvature and torsion are defined by the connection appearing in:

$$[D_\mu, D_\nu]_{HD} V^\rho := (D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho))_{HD}. \quad (8.22)$$

Using Eq. (8.21) in Eq. (8.9) it follows that:

$$\tilde{R}^\lambda_{\rho\mu\nu} := (\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho})_{HD}, \quad (8.23a)$$

$$\tilde{T}^\lambda_{\mu\nu} := (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})_{HD}, \quad (8.23b)$$

and it also follows that:

$$D \wedge \tilde{T} := \tilde{R} \wedge q. \quad (8.24)$$

This equation in tensor notation is:

$$\begin{aligned} & \partial_\mu \tilde{T}^a_{\nu\rho} + \partial_\rho \tilde{T}^a_{\mu\nu} + \partial_\nu \tilde{T}^a_{\rho\mu} + \omega^a_{\mu b} \tilde{T}^b_{\nu\rho} + \omega^a_{\rho b} \tilde{T}^b_{\mu\nu} + \omega^a_{\nu b} \tilde{T}^b_{\rho\mu} \\ & := \tilde{R}^a_{\mu\nu\rho} + \tilde{R}^a_{\rho\mu\nu} + \tilde{R}^a_{\nu\rho\mu} \end{aligned} \quad (8.25)$$

and this sum is equivalent to:

$$D_\mu T^{a\mu\nu} := R^a_{\mu}{}^{\mu\nu}. \quad (8.26)$$

A particular solution of Eq. (8.26) is:

$$D_\mu T^{\kappa\mu\nu} := R^\kappa_{\mu}{}^{\mu\nu}. \quad (8.27)$$

This equation is the most convenient form of the Bianchi identity and shows that the covariant derivative of the torsion tensor on the left hand side is identically equal to the Ricci type tensor appearing on the right hand side of Eq. (8.26). It has been shown by computer algebra for many different metrics that the tensor $R^\kappa_{\mu}{}^{\mu\nu}$ is in general non-zero for a Christoffel connection:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}. \quad (8.28)$$

However, for the connection (8.28) the torsion must be zero:

$$T_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda} = 0. \quad (8.29)$$

Therefore the Bianchi identity and Christoffel connection are incompatible in general. The Christoffel connection can be a solution of the Bianchi identity if and only if the metric defines a Ricci flat space-time where all elements of the Ricci tensor vanish [1–12]. However, for such a space-time the EH field equation means that the $T_{\mu\nu}$ tensor also vanishes. This is the canonical energy-momentum density tensor as is well known. Crothers has shown [13] that the Ricci flat space-time is incompatible with the Einstein equivalence principle and in such a space-time there can be no fields and no mass/energy density. Such a space-time has no physical meaning therefore. Several other conceptual errors in the EH theory have been pointed out by Crothers [13, 14].

There is no way in which the EH theory can escape its inherent weaknesses. Furthermore it is almost entirely based on the use of the Christoffel symbol, which as we have shown, violates the fundamental geometry (8.1). It is concluded that all cosmologies based on EH theory are physically meaningless, notably Big Bang, black hole theory and dark matter. None can exist in nature. The Christoffel symbol can be used if and only if

$$R^{\kappa}_{\mu}{}^{\mu\nu} = 0 \quad (8.30)$$

when Eq. (8.1) reduces to:

$$R \wedge q = 0 \quad (8.31)$$

This in tensor notation is:

$$R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} + R^{\lambda}_{\mu\nu\rho} = 0 \quad (8.32)$$

and is incorrectly referred to as the “first Bianchi identity”. In fact it was first given by Ricci and Levi-Civita and later developed by Cartan into Eq. (8.1). The so called “second Bianchi identity” is in form notation:

$$D \wedge R = 0 \quad (8.33)$$

but it has been shown during the course of the development of ECE theory [1–11] that the “second Bianchi identity” is a restatement of Eq. (8.1) as follows:

$$D \wedge (D \wedge T) := D \wedge (R \wedge q). \quad (8.34)$$

So there is only one Bianchi identity, Eq. (8.1).

The incompatibility of the Christoffel connection with Eq. (8.1) was discovered in paper 93 of the ECE series (www.aias.us) when it was found that the tensor $R^\kappa{}_\mu{}^{\mu\nu}$ is non-zero in general for a Christoffel connection, and zero only for physically meaningless Ricci flat space-times. This property of the Christoffel connection is incompatible with the fact that for such a connection, the torsion tensor must be zero. The Christoffel connection cannot be a solution of Eq. (8.27) in general. To calculate $R^\kappa{}_\mu{}^{\mu\nu}$ required computer algebra, because the calculation of $R^\kappa{}_\mu{}^{\mu\nu}$ by hand is exceedingly intricate. So this fatal flaw in EH theory has remained hidden for more than ninety years. The EH equation itself is valid if and only if it does not use a Christoffel connection and does not use a Ricci flat space-time. However, ECE shows that the EH field equation is only one part of the whole.

8.3 Interaction of Gravitation and Electromagnetism

It has been shown in Section 8.2 that the interpretation of the well known EH theory with a Christoffel connection is basically incorrect geometrically. This may perhaps be surprising to the uninitiated, but it has been known for some years to scholars like Crothers [13–14] for example that EH is deeply flawed in several ways. The dogma of EH is unfortunately adhered to in the cosmological literature, and this has become harmful to science. In this section the basic equation of paper 93 of www.aias.us is derived by considering Eq. (8.27) for the interaction of two fields. It is found that in this case the calculation of $R^\kappa{}_\mu{}^{\mu\nu}$ with the Christoffel symbol can be approximately valid, but if and only if one field interacts with another in the context of ECE theory.

The ECE theory asserts that the electromagnetic field tensor is the Cartan torsion within a proportionality:

$$F_{\mu\nu}^a = A^{(0)} T_{\mu\nu}^a \quad (8.35)$$

so there exists an electromagnetic torsion, denoted T (e/m), as well as a gravitational torsion denoted T (grav). Similarly there exists an electromagnetic curvature R (e / m) as well as a gravitational curvature R (grav). The connection in ECE theory is not the symmetric Christoffel connection. The electromagnetic field in ECE theory is for all practical purposes defined by the balance condition:

$$R \wedge q = \omega \wedge T \quad (8.36)$$

which is based on experimental data, notably that the homogenous field equation must be:

$$d \wedge F^a = 0 \quad (8.37)$$

for all practical purposes in the laboratory. The condition (8.36) then means that there is no interaction between gravitation and electromagnetism in this well defined sense. From Eq. (8.36) it follows that:

$$\tilde{R} \wedge q = \omega \wedge \tilde{T} \quad (8.38)$$

so that there is no contribution from electromagnetism to the charge current density of the inhomogeneous field equation [1–10] of ECE. Under these circumstances the basic equation of paper 93 follows:

$$\partial_\mu F^{\kappa\mu\nu}(e/m) \doteq A^{(0)} R^\kappa{}_\mu{}^{\mu\nu}(\text{grav}) \quad (8.39)$$

where the electromagnetic torsion on the left hand side is balanced by the gravitational curvature on the right hand side. Under these circumstances the Christoffel connection can be used to compute $R^\kappa{}_\mu{}^{\mu\nu}$, and this automatically implies that there is no gravitational torsion present.

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